hep-th/9408080

The Dynamical Algebra of the Hydrogen Atom as a Twisted Loop Algebra[1]

Claudia Daboul *, Jamil Daboul ♯, Peter Slodowy*

*Mathematisches Seminar, Universität Hamburg
Bundesstr. 55, D-2000 Hamburg 13, Germany

♯Physics Department, Ben Gurion University of the Negev
84105 Beer Sheva, Israel (e-mail: daboul@bguvms.bgu.ac.il)

Abstract

We show that the dynamical symmetry of the hydrogen atom leads in a natural way to an infinite-dimensional algebra, which we identify as the positive subalgebras of twisted Kac-Moody algebras of $so(4)$. We also generalize our results to the N-dimensional hydrogen atom. For odd N, we identify the dynamical algebra with the positive part of the twisted algebras $\hat{so}(N+1)^\tau$. However, for even N this algebra corresponds to a parabolic subalgebra of the untwisted loop algebra $\hat{so}(N+1)$.

1 Introduction

It is well known that in the Kepler problem, defined by the Hamiltonian $H = \boldsymbol{p}^2/2\mu - \alpha/r$, the **Runge-Lenz vector**[1−2]

$$\boldsymbol{A} = \frac{1}{2}[\boldsymbol{p}\times\boldsymbol{L} - \boldsymbol{L}\times\boldsymbol{p}] - \mu\alpha\frac{\boldsymbol{r}}{r} , \tag{1}$$

is conserved, $[H, \boldsymbol{A}] = 0$. Here, μ is the (reduced) mass, and α is any coupling constant, which for the Hydrogen atom is equal to e^2.

The components of $\boldsymbol{A}$ and the angular momentum vector $\boldsymbol{L}$ have the following commutation relations :

$$[L^i, L^j] = i\hbar\epsilon_{ijk}L^k , \quad [L^i, A^j] = i\hbar\epsilon_{ijk}A^k , \quad [A^i, A^j] = i\hbar\epsilon_{ijk}(-2\mu H)L^k . \tag{2}$$

The 6 operators L^i and A^i do *not* form a closed finite-dimensional algebra on the whole Hilbert space $\mathcal{H}$, because the Hamiltonian H appears on the r.h.s. of (2). Therefore, in the standard treatments one concentrates on individual subspaces $\mathcal{H}(E)$ which belong to definite energies E. In each such subspace, the Hamiltonian in (2) can be replaced by its eigenvalue E. This led people to identify the dynamical algebra with three different algebras, namely $so(4)$, $e(3)$ and $so(3,1)$, depending on the value of the energy, as I shall explain later on. This situation is not satisfactory, since the identification of the algebra should not depend of the energy.

I shall now show that the dynamical algebra of the Kepler problem can be identified in a natural way with the infinite dimensional twisted loop algebra of $so(4)$, and then give a few comments on the generalization to the formalism to N-dimensional Hydrogen atom.

[1] Based on a talk given by J. Daboul at the XX International Colloquium on "Group Theoretical Methods in Physics", Osaka, July 3-9, 1994

2 The standard identification of the dynamical algebra

Let me first recall that each of the three algebras $so(4)$, $e(3)$ and $so(3,1)$ is defined in terms of 6 generators, which can be written as two 3-vectors: $\boldsymbol{J}$ and $\boldsymbol{M}^{\eta}$, which obey the following commutation relations :

$$[J^i, J^j] = i\epsilon_{ijk} J^k \ , \quad [J^i, M^{\eta,j}] = i\epsilon_{ijk} M^{\eta,k} \ , \quad [M^{\eta,j}, M^{\eta,j}] = \eta i\epsilon_{ijk} J^k \ , \tag{3}$$

where the summation over the repeated index k is implied. For $\eta = 1, 0, -1$, the above commutation relations define the three algebras $so(4)$, $e(3)$ and $so(3,1)$, respectively. Note that I am using J^i instead of L^i in the abstract definition of the algebras (3), in order to distinguish between the J^i and their differential-operator representations L^i.

As I said before, in the usual treatment one concentrates on individual subspaces $\mathcal{H}(E)$ which belong to definite energies E. For each such subspace, one can replace the Hamiltonian in (2) by its eigenvalue E. One then normalizes A^i, and obtains algebras, which are isomorphic to the three given in (3).

For example, for negative energies, the spectrum is discrete (E_n with $n = 1, 2, \ldots$). Here, one usually defines "normalized" Runge-Lenz vectors by, $\tilde{\boldsymbol{A}}(E_n) := \boldsymbol{A}/\sqrt{-2\mu E_n}$, which lead to commutation relations similar to those of $so(4)$, $[\tilde{\boldsymbol{A}}^i(E_n), \tilde{\boldsymbol{A}}^j(E_n)] = i\hbar\epsilon_{ijk} L^k$. Since the energy subspaces $\mathcal{H}(E_n)$ have n^2 degenerate levels, the above procedure leads to $n^2 \times n^2$ irreducible matrix representations of the operators L^i and $\tilde{A}^i(E_n)$ and thus of $so(4)$.

For the positive spectrum, one defines $\tilde{\boldsymbol{A}}(E) := \boldsymbol{A}/\sqrt{2\mu E}$, so that $[\tilde{\boldsymbol{A}}^i(E), \tilde{\boldsymbol{A}}^j(E)] = -i\hbar\epsilon_{ijk} L^k$. In this way, one gets for every $E > 0$ a different representation of $so(3,1)$ in terms of automorphisms (differential operators) on the subspace $\mathcal{H}(E)$.

Finally, for $E = 0$ there is no need for normalization, since the A^i commute among themselves, so that L^i and A^i, when applied to $\mathcal{H}(E = 0)$ leads automatically to an infinite dimensional representation of the Euclidean algebra $e(3)$.

3 The Infinite-Dimensional H-algebra $I\!\!H$

In our new treatment, we keep the operators A^i as they are. Instead, we include in the algebra all the products of L^i and A^i with the *positive* powers of the Hamiltonian H. Thus, we define

$$L^i_n := \hat{H}^n L^i \quad \text{and} \quad A^i_n := \hat{H}^n A^i \ , \ \text{where } n \geq 0 \ , \ i = 1,2,3, \ \text{and } \hat{H} := -2\mu H \ . \tag{4}$$

In this way, we obtain a closed but infinite-dimensional algebra, which we shall call the **H-algebra** and denote it by $I\!\!H$. The commutation relations follow immediately from (2)

$$[L^i_n, L^j_m] = i\hbar\epsilon_{ijk} L^k_{n+m} \ , \quad [L^i_n, A^j_m] = i\hbar\epsilon_{ijk} A^k_{n+m} \ , \quad [A^i_n, A^j_m] = i\hbar\epsilon_{ijk} L^k_{n+m+1} \ . \tag{5}$$

This algebra looks exactly like the loop algebra of $so(4)$, *except for the extra 1 in the lower index of* L^k_{n+m+1}. Because of this extra 1, the H-algebra turns out to be isomorphic to the positive part of the *twisted* loop algebra of $so(4)$, as we shall show below.

3.1 Quotient algebras of $I\!\!H$: A formal construction

Even before identifying $I\!\!H$ I'll now show how we can formally reproduce the three algebras $so(4)$, $so(3,1)$ and $e(3)$, as quotient algebras of $I\!\!H$, by using the following construction: Clearly, $I(c) :=$

$(\hat{H}-c)H\!\!I$ is an ideal of $H\!\!I$ for every real parameter c. Therefore, the quotient algebra $H\!\!I/I(c)$ has only 6 basis elements, which can be represented by L^i and A^i. The elements are the subspaces $\hat{L}^i \equiv L^i + I(c)$ and $\hat{A}^i \equiv A^i + I(c)$. By recalling that in the quotient algebra the ideal $I(c)$ acts as the zero element, we easily get the following commutation relations:

$$[\hat{L}^i, \hat{L}^j] = i\hbar\epsilon_{ijk}\hat{L}^k \ , \quad [\hat{L}^i, \hat{A}^j] = i\hbar\epsilon_{ijk}\hat{A}^k \ , \quad [\hat{A}^i, \hat{A}^j] = \hat{H}\ i\hbar\epsilon_{ijk}\hat{L}^k = c\ i\hbar\epsilon_{ijk}\hat{L}^k \ , \tag{6}$$

which are similar to (3). Therefore, the commutation relations (6) define algebras which are isomorphic to $so(4)$, $so(3,1)$ and $e(3)$, for $c>0, c<0$, and $c=0$, respectively. The above construction can be summarized, as follows:

$$Q(c) \equiv H\!\!I/I(c) \simeq \begin{cases} so(4), & \text{for} \quad c>0, \\ so(3,1), & \text{for} \quad c<0, \\ e(3), & \text{for} \quad c=0. \end{cases} \tag{7}$$

The use of the ideal $(\hat{H}-c)H\!\!I$ is practically equivalent to the usual projection procedure on the eigenspaces $\mathcal{H}(E)$, if $c=-2\mu E$.

4 The standard and the twisted Kac-Moody algebras of so(4)

A short review of the basic notions of the standard and the twisted affine Kac-Moody algebras was given recently by us[3]. For more general expositions I refer to references[4,5].

Here, I shall only give the definitions for the specific loop algebras of $so(4)$ and its twisted counterpart: The loop algebra of $so(4)$ is obtained by taking infinitely many copies of the 6 original generators. These copies are distinguished by a lower index $n \in Z$. Thus the loop algebra is generated by the following set of elements (From now on we shall use M^i instead of $M^{\eta=1,i}$, which was defined in Eq. (3)):

$$\hat{\mathcal{G}} := \{J^i_n\} \cup \{M^i_n\} \ , \qquad \text{where} \quad i=1,2,3, \qquad \text{and} \quad n \in Z \tag{8}$$

The commutation relations among these are:

$$[J^i_m, J^i_n] = i\epsilon_{ijk}J^k_{m+n} \ , \quad [J^i_m, M^j_n] = i\epsilon_{ijk}M^k_{m+n} \ , \quad [M^i_m, M^j_n] = i\epsilon_{ijk}J^k_{n+m} \ . \tag{9}$$

It is easy to see that the following subset of $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}}^\tau := \{J^i_{2n}\} \cup \{M^i_{2n+1}\} \subset \hat{\mathcal{G}} \ , \qquad \text{where} \quad i=1,2,3, \qquad \text{and} \quad n \in Z \tag{10}$$

form a *subalgebra* of $\hat{\mathcal{G}}$:

$$[J^i_{2m}, J^j_{2n}] = i\epsilon_{ijk}J^k_{2m+2n} \ , \quad [J^i_{2m}, M^j_{2n+1}] = i\epsilon_{ijk}M^k_{2m+2n+1} \ , \quad [M^i_{2m+1}, M^j_{2n+1}] = i\epsilon_{ijk}J^k_{2(n+m+1)} \ . \tag{11}$$

This subalgebra is called the **twisted Loop algebra** of $\hat{so}(4)$. The τ denotes the involution automorphism, which is needed to define the twisting. This is explained in Ref.[3].

To get the Kac-Moody algebra from the corresponding loop algebra, one has to modify the above commutation relation by adding to the right-hand sides terms that are proportional to $\mathcal{K}$, which is an operator which commutes with all the generators T^a_n, and is called the **central element**. In our case, the central term will be identically zero. For this reason the loop algebra is sometimes called "centerless Kac-Moody algebra".

5 Identification of the H-Algebra with Twisted Loop Algebras

It is easy to check that the following map from the abstract positive subalgebra $\mathcal{P} \equiv \hat{so}(4)^\tau_+$ of $\hat{so}(4)^\tau$ onto the H-algebra $I\!\!H$ (5),

$$\varphi : \ \mathcal{P} \longmapsto I\!\!H \ , \quad \text{where} \quad \varphi(J^i_{2n}) = \frac{1}{\hbar} L^i_n \ , \qquad \text{and} \quad \varphi(M^i_{2n+1}) = \frac{1}{\hbar} A^i_n \ , \quad n \geq 0 \ , \tag{12}$$

is a homomorphism. For example,

$$\begin{aligned} [\varphi(M^i_{2m+1}), \varphi(M^j_{2n+1})] &= \frac{1}{\hbar^2}[A^i_m, A^j_n] = i\frac{1}{\hbar}\epsilon_{ijk} L^k_{m+n+1} \\ &= i\epsilon_{ijk}\varphi(J^k_{2(m+n+1)}) = \varphi([M^i_{2m+1}, M^j_{2n+1}]) \ , \end{aligned} \tag{13}$$

where we used the commutation relations (5).

This map defines a representation of $\mathcal{P}$ in terms of the dynamical operators H, L^i and A^i of the hydrogen atom. In fact, the map (12) is an *isomorphism* between $\mathcal{P}$ and $I\!\!H$. This is because H has an infinite number of different eigenvalues, which insures that the images of different $T^a_n \in \mathcal{P}$ are linearly independent.

6 Conclusions and Outlook

Since the famous paper of Pauli in 1926 on the energy levels of the H-atom, numerous papers have been written on the symmetry of the Hydrogen atom. I believe that we have now given the first correct identification of the dynamical algebra of the H-atom.

We also generalized the whole formalism to the N-dimensional hydrogen atom [6], and found that for odd N the dynamical algebra is the positive subalgebra of the twisted algebra, $\hat{so}(N+1)^\tau$, as expected. However, for even N the twisted algebra can be untwisted, so that the dynamical algebra is a *parabolic subalgebra* of the (untwisted) loop algebra $\hat{so}(N+1)$. We hope to publish the details soon.

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